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# TYPES OF ACYCLICITY *

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## 1. Introduction

My main purpose in this note is to prove:

**1.1. Theorem.** *If $E_*$ is a connective homology theory, then $E_*$ has the same acyclic spaces as $H_*(\cdot; A)$, where either $A = \mathbf{Z}[J^{-1}]$ or $A = \oplus_{p\in J}\mathbf{Z}/p\mathbf{Z}$ for some set $J$ of primes.*

By $\mathbf{Z}[J^{-1}]$, I mean the rationals whose denominators are products of primes in $J$. By a *homology theory*, I mean a generalized homology theory defined on arbitrary CW pairs and satisfying the direct limit axiom [2, p. 188]. I say that a homology theory $F_*$ is *connective* if $F_i(pt) = 0$ for $i$ sufficiently small; and I say that two homology theories $F_*$ and $G_*$ have the same acyclic spaces if the following equivalent conditions hold:

(i) For a CW-complex $X$, $\widetilde{F}_*(X) = 0$ if and only if $\widetilde{G}_*(X) = 0$.

(ii) For a map $f: X \to Y$ of CW-complexes, $f_*: F_*(X) \to F_*(Y)$ is an isomorphism if and only if $f_*: G_*(X) \to G_*(Y)$ is an isomorphism.

The proof of 1.1 is in Section 4, where I will also prove an analogous result for a cohomology theory $E^*$ with $E^i(pt) = 0$ for $i$ sufficiently small.

It should not be surprising that there are so few types of acyclicity under our assumptions. Indeed, results of Adams [1, §14] implied 1.1 for many connective multiplicative homology theories; and results of Bousfield and Kan [5] implied 1.1 for homology with coefficients in any commutative ring.

Perhaps the main interest of 1.1 lies in the theory of localizations of spaces. Each homology theory determines a localization functor which selects a canonical homotopy type within each homology equivalence class of spaces (see [4]); and 1.1 provides a partial inventory of the distinct localization functors.

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## 2. Special classes of abelian groups

As a step toward proving 1.1, I will study the following algebraic notion.

**2.1. Definition.** A class $\mathcal{M}$ of abelian groups is *special* if it satisfies:
(i) $0 \in \mathcal{M}$.
(ii) $\mathcal{M}$ is closed under arbitrary direct sums.
(iii) If $A_1 \to A_2 \to A_3 \to A_4 \to A_5$ is an exact sequence of abelian groups with $A_1, A_2, A_4, A_5 \in \mathcal{M}$, then $A_3 \in \mathcal{M}$.

**2.2. Example.** For an abelian group $A$, one can obtain a special class by taking all $G$ with $G \otimes A = 0 = \mathrm{Tor}(G, A)$, or by taking all $G$ with $\mathrm{Hom}(G, A) = 0 = \mathrm{Ext}(G, A)$.

There are very few special classes.

**2.3. Proposition.** *The only special classes of abelian groups are the J-primary torsion groups and the uniquely J-divisible groups, where J is a set of primes.*

I will deduce 2.3 from the following lemmas concerning a special class $\mathcal{M}$ of abelian groups.

**2.4. Lemma.** *$\mathcal{M}$ is closed under direct limits.*

**Proof.** This holds since $\mathcal{M}$ is closed under sums and cokernels.

**2.5. Lemma.** *If $\mathbf{Z}/p\mathbf{Z} \in \mathcal{M}$ for each p in a set J of primes, then every J-primary torsion group is in $\mathcal{M}$.*

**Proof.** This follows by 2.4 since the finitely generated $J$-primary torsion groups are clearly in $\mathcal{M}$.

**2.6. Lemma.** *If $\mathbf{Q}$ (= the rationals) is in $\mathcal{M}$ and $\mathbf{Z}/p\mathbf{Z} \in \mathcal{M}$ for each prime p not in a set J, then every uniquely J-divisible group is in $\mathcal{M}$.*

**Proof.** Since $\mathbf{Q}/\mathbf{Z}[J^{-1}] \in \mathcal{M}$ by 2.5, it follows that $\mathbf{Z}[J^{-1}] \in \mathcal{M}$; and thus the finitely generated $\mathbf{Z}[J^{-1}]$-modules are in $\mathcal{M}$. The lemma now follows by 2.4.

**2.7. Lemma.** *If some $G \in \mathcal{M}$ is not uniquely p-divisible for p prime, then $\mathbf{Z}/p\mathbf{Z} \in \mathcal{M}$.*

**Proof.** This follows since $\mathcal{M}$ contains a non-zero $\mathbf{Z}/p\mathbf{Z}$-module given by the kernel or cokernel of $G \xrightarrow{p} G$.

**2.8. Lemma.** *If some $G \in \mathcal{M}$ is not a torsion group, then $\mathbf{Q} \in \mathcal{M}$.*

**Proof.** This follows since $\mathcal{M}$ contains the non-zero **Q**-module $\mathbf{Q} \otimes G$, which is the direct limit of $G \xrightarrow{2} G \xrightarrow{3} G \xrightarrow{4} \dots$ .

**2.9. Proof of 2.3.** For a special class $\mathcal{M}$, the above lemmas imply the following refinement of 2.3:

(i) If $\mathbf{Q} \notin \mathcal{M}$ and $\mathbf{Z}/p\mathbf{Z} \in \mathcal{M}$ exactly for $p \in J$, then $\mathcal{M}$ consists of the $J$-primary torsion groups.

(ii) If $\mathbf{Q} \in \mathcal{M}$ and $\mathbf{Z}/p\mathbf{Z} \in \mathcal{M}$ exactly for $p \notin J$, then $\mathcal{M}$ consists of the uniquely $J$-divisible groups.

## 3. Special classes of connective spectra

I will now carry the results of Section 2 into the stable homotopy category (see e.g. [1]). Recall that a spectrum $E$ is *connective* if $\pi_i E = 0$ for $i$ sufficiently small.

**3.1. Definition.** A class $\mathcal{N}$ of connective spectra is *special* if:

(i) $\mathcal{N}$ contains the trivial spectrum.

(ii) $\mathcal{N}$ contains a coproduct $\mathbf{V}_\alpha X_\alpha$ of connective spectra if and only if $\mathcal{N}$ contains each $X_\alpha$ and the coproduct is connective.

(iii) In a cofibre triangle of connective spectra, if $\mathcal{N}$ contains two of the spectra then it contains the third.

(iv) If $Y \in \mathcal{N}$, then $X \wedge Y \in \mathcal{N}$ for each connective spectrum $X$.

(v) If each Postnikov section of a connective spectrum $X$ is in $\mathcal{N}$, then $X \in \mathcal{N}$.

I will show that the special classes of connective spectra correspond to the special classes of abelian groups.

**3.2. Example.** For a special class $\mathcal{M}$ of abelian groups, let $S(\mathcal{M})$ be the class consisting of those connective spectra $X$ such that $\pi_i X \in \mathcal{M}$ for all $i$. Then $S(\mathcal{M})$ is clearly a special class. Moreover, it is easy to show (using 2.3 and $\mathcal{C}$-theory) that a connective spectrum $X$ is in $S(\mathcal{M})$ if and only if $H_i(X; \mathbf{Z}) \in \mathcal{M}$ for all $i$.

**3.3. Proposition.** *Any special class $\mathcal{N}$ of connective spectra is of the form $S(\mathcal{M})$, where $\mathcal{M}$ is a special class of abelian groups.*

**Proof.** Let $\mathcal{M}$ consist of those abelian groups $G$ such that $HG \in \mathcal{N}$, where $HG$ denotes the Eilenberg–MacLane spectrum with group $G$. Then $\mathcal{M}$ is clearly a special class, and $S(\mathcal{M}) \subset \mathcal{N}$ by 3.1 (iii) and (v). To show $\mathcal{N} \subset S(\mathcal{M})$, suppose $Y \in \mathcal{N}$. Then $H\mathbf{Z} \wedge Y \in \mathcal{N}$ by 3.1 (iv), and

$$H\mathbf{Z} \wedge Y \simeq \bigvee_i S^i \wedge HG_i$$

by [1, p. 227], where $G_i = H_i(Y; \mathbf{Z})$. Thus $H_i(Y; \mathbf{Z}) \in \mathcal{M}$ for all $i$, and the Hurewicz theorem implies that the lowest non-trivial homotopy group of $Y$ is in $\mathcal{M}$. An easy inductive argument now shows that the Eilenberg sections of $Y$ are in $\mathcal{N}$ and that their lowest non-trivial homotopy groups are in $\mathcal{M}$. Hence $Y \in S(\mathcal{M})$.

For future reference, I will give:

**3.4.** *Criteria for identifying special classes.* Let $\mathcal{U}$ (resp. $\mathcal{V}$) be the set consisting of the Eilenberg–MacLane (resp. Moore) spectra of type $\mathbf{Q}$ and of type $\mathbf{Z}/p\mathbf{Z}$ for all primes $p$. *If $\mathcal{N}$ and $\mathcal{N}'$ are special classes of connective spectra with $\mathcal{N} \cap \mathcal{U} = \mathcal{N}' \cap \mathcal{U}$ or $\mathcal{N} \cap \mathcal{V} = \mathcal{N}' \cap \mathcal{V}$, then $\mathcal{N} = \mathcal{N}'$.* This follows easily from 2.9, 3.2 and 3.3.

## 4. Results on acyclicity

In this section, I will prove a general acyclicity theorem (4.5) which implies both 1.1 and an analogous result (4.7) for cohomology.

Recall that a spectrum $E$ determines a homology functor $E_*$ and a cohomology functor $E^*$ on spectra (see e.g. [1, §6]). Let $\mathcal{N}_E$ (resp. $\mathcal{N}^E$) denote the class of connective spectra $X$ such that $E_*(X) = 0$ (resp. $E^*(X) = 0$). It is easy to show:

**4.1. Lemma.** *For any spectrum $E$, $\mathcal{N}_E$ and $\mathcal{N}^E$ satisfy all axioms for a special class except* 3.1(v).

Unfortunately 3.1 (v) may be violated. For instance:

**4.2. Example.** Let $E$ be the BU spectrum. Then $\mathcal{N}^E$ contains $HG$ for each finite torsion group $G$ by [3], although $\mathcal{N}^E$ clearly does not contain the Moore spectrum of type $\mathbf{Z}/2\mathbf{Z}$. Thus $\mathcal{N}^E$ cannot be a special class, since it is not of the form $S(\mathcal{M})$.

To get around this difficulty, call a spectrum $E$ *anti-connective* if $\pi_i E = 0$ for $i$ sufficiently large.

**4.3. Lemma.** *If $E$ is a connective spectrum, then $\mathcal{N}_E$ is a special class. If $E$ is an anti-connective spectrum, then $\mathcal{N}^E$ is a special class.*

**Proof.** This follows by 4.1 and an Atiyah–Hirzebruch spectral sequence argument.

In order to identify the special classes $\mathcal{N}^E$, I will need the notion of:

**4.4.** *Ext-complete groups.* An abelian group $B$ is *Ext-complete* if $\operatorname{Hom}(\mathbf{Q}, B) = 0 = \operatorname{Ext}(\mathbf{Q}, B)$ or, equivalently, if the canonical homomorphism $B \to \operatorname{Ext}(\mathbf{Q}/\mathbf{Z}, B)$ is an isomorphism. The Ext-complete groups have been completely classified by Harrison

[7] (see also [6, pp. 165–182]). Some examples of Ext-complete groups are the finite groups and their inverse limits.

I can now give my strongest result on acyclicity.

Let $E$ be a spectrum; let $I$ be the set of primes $p$ such that each $\pi_i E$ is uniquely $p$-divisible; and let $J$ be the set of primes not in $I$.

**4.5. Theorem.** (i) *If $E$ is connective with each $\pi_i E$ a torsion group, then* $\mathcal{N}_E = \mathcal{N}_{HA}$ *for* $A = \oplus_{p\in J}\mathbf{Z}/p\mathbf{Z}$.

(ii) *If $E$ is connective with some $\pi_i E$ not a torsion group, then* $\mathcal{N}_E = \mathcal{N}_{HA}$ *for* $A = \mathbf{Z}[I^{-1}]$.

(iii) *If $E$ is anti-connective with each $\pi_i E$ Ext-complete, then* $\mathcal{N}^E = \mathcal{N}_{HA}$ *for* $A = \oplus_{p\in J}\mathbf{Z}/p\mathbf{Z}$.

(iv) *If $E$ is anti-connective with some $\pi_i E$ not Ext-complete, then* $\mathcal{N}^E = \mathcal{N}_{HA}$ *for* $A = \mathbf{Z}[I^{-1}]$.

**Proof.** The above classes $\mathcal{N}_E$, $\mathcal{N}^E$ and $\mathcal{N}_{HA}$ are all special by 4.3; and the desired equalities follow from the Moore version of 3.4.

I can now give:

**4.6. Proof of Theorem 1.1.** By [2, p. 188], the given homology theory $E_*$ must arise from some spectrum $E$. Thus 1.1 follows easily from 4.5.

In the same way, one shows:

**4.7. Theorem.** *If $E^*$ is an anti-connective cohomology theory, then $E^*$ has the same acyclic spaces as $H_*(-;A)$, where either $A = \mathbf{Z}[J^{-1}]$ or $A = \oplus_{p\in J}\mathbf{Z}/p\mathbf{Z}$ for some set $J$ of primes.*

By a *cohomology theory*, I mean a generalized cohomology theory defined on CW pairs and satisfying the wedge axiom. I say that a cohomology theory $F^*$ is *anti-connective* if $F^i(pt) = 0$ for $i$ sufficiently small.

**4.8. Remark.** In 1.1 (resp. 4.7) the required group $A$ depends only on the coefficients $\pi_i E = E_i(pt)$ (resp. $\pi_{-i}E = E^i(pt)$). Indeed 4.5 gives a recipe for computing $A$ from these coefficients.

**4.9. Remark.** Clearly $H_*(-;\mathbf{Z}[J^{-1}])$ has the same acyclic spaces as $H^*(-;\mathbf{Z}[J^{-1}])$; and $H_*(-;\oplus_{p\in J}\mathbf{Z}/p\mathbf{Z})$ has the same acyclic spaces as $H^*(-;\prod_{p\in J}\mathbf{Z}/p\mathbf{Z})$. Moreover, it is now evident that the connective homology theories provide exactly the same types of acyclicity as the anti-connective cohomology theories.

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